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The effect of mutual interference between predators on a predator–prey model with diffusion ☆

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ABSTRACT

We consider a diffusive predator–prey model with Beddington–DeAngelis functional response under homogeneous Dirichlet boundary conditions. The effect of large k which represents the extent of mutual interference between predators is extensively studied. By making use of the fixed point index theory, we obtain a complete understanding of the existence, uniqueness and stability of positive steady-states when k is sufficiently large. Moreover, we present some numerical simulations that supplement the analytic results in one dimension.

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1. Introduction

In this paper, we are concerned with the following diffusive predator–prey model

$$\begin{cases} u_t - \Delta u = \left(a - u - \dfrac{bv}{1+mu+kv}\right)u, & x \in \Omega,\ t > 0, \\ v_t - \Delta v = \left(c - v + \dfrac{du}{1+mu+kv}\right)v, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_0(x)$ and $v_0(x)$ are non-negative continuous functions and not identically zero. Ω is a bounded domain in R^N with sufficiently smooth boundary $\partial\Omega$. u and v represent the densities of the prey and the predator, respectively. The parameters a, b, c, d, m and k are constants with a, b, d positive and m, k non-negative; c may change sign. The corresponding ordinary differential system was introduced by Beddington [1] and DeAngelis et al. [2].

In model (1.1), $u/(1+mu+kv)$ is the so-called Beddington–DeAngelis (simply write as B–D) functional response. Compared to Holling–Tanner functional response, it has an extra term "kv" in the denominator which models mutual interference between predators. In addition, B–D functional response represents most of qualitative features of the ratio-

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dependent model, but avoids the "low density" problem which is usually the source of controversy. For more detailed biological implications, the interested reader may further refer to [3] and references therein.

Studies on the steady-states in reaction–diffusion system are the hot point question all along. Our paper also deals with the steady-state problem corresponding to (1.1), which takes in the following form

$$\begin{cases} -\Delta u = \left(a - u - \dfrac{bv}{1+mu+kv}\right)u, & x \in \Omega, \\ -\Delta v = \left(c - v + \dfrac{du}{1+mu+kv}\right)v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

If $k, m = 0$, then (1.2) is reduced to the classical Lotka–Volterra predator–prey model which has received extensive studies in the past several decades, see [4–6]. If $k = 0$ and $m > 0$, then (1.2) is a predator–prey model of Holling–Tanner type. Blat and Brown in [7] obtained the existence of positive solutions by making use of local and global bifurcation theories. In [8], problem (1.2) with $k = 0$ was also studied and sufficient conditions for the existence of positive solutions were gained. It was observed in [8] that these sufficient conditions are not necessary if m is large. The case when m goes to infinity was extensively studied by Du and Lou in [9–11]. They gave a good understanding of the existence, stability and number of positive solutions for large m.

If $k, m > 0$, then (1.2) is a predator–prey model of B–D type. To our knowledge, there are not many works on such type of functional response in the reaction–diffusion system. It should be noted that under Neumann boundary conditions, Chen and Wang established the existence of non-constant positive solutions using topological degree theory, see [12]. Recently, under Dirichlet boundary conditions we gave some multiplicity and uniqueness results in [13].

Since the parameter k represents the mutual interference between predators, then in this paper, we are mainly concerned with positive solutions of (1.2) in the case that k is large. More precisely, we want to know the effect of large k on positive solutions. In fact, when k is large, we have established the uniqueness of positive solutions in the case that $c > \lambda_1$ and c is bounded away from λ_1, see Theorem 3.1 in [13]. Obviously, the proof there only deals with the simplest case. Our discussion below will involve several different and more complicated cases, which are characterized by different asymptotic behaviors of positive solutions when k goes to ∞, and we finally show that for any $c \in R$, (1.2) has at most one positive solution if k is large enough.

However, it should be noted that the uniqueness of positive solutions does not necessarily need k sufficiently large. In [14], a range of parameters for the uniqueness of positive solutions is described in one dimension. For the case of N dimensions, the method in [14] does not work any more. But, by the fixed point index theory, we find that when a is close to λ_1 and $c \leqslant \lambda_1$, (1.2) has at most one positive solution for any $k \geqslant 0$. Furthermore, we establish a more general result when a is close to λ_1 and k is bounded.

Throughout this paper, we always reduce the proof of uniqueness and stability to the proof of the fact that any possible positive solution is non-degenerate and linearly stable. This is a widely used trick and often involves some estimates and indirect arguments, see [9–11].

This paper is organized as follows. In Section 2, we give some known results, which include the existence and non-existence of positive solutions to (1.2). In Section 3, we obtain the uniqueness of positive solutions to (1.2) for any $c \in R$ in the case that k is large. Due to different asymptotic behaviors of positive solutions, we divide our discussion into four cases and obtain Theorems 3.1, 3.2, 3.3 and 3.4, respectively. Moreover, we find that the uniqueness of positive solutions does not necessarily need k sufficiently large when c is large or when a is close to λ_1. In Section 4, we present some numerical simulations that supplement the analytic results in one dimension.

2. Preliminaries

In this section, we give some known results, which include the existence and non-existence of positive solutions to (1.2). One can refer to [13] for detailed proofs. First, we need to give some notations and basic facts which will be often used later.

Let $\lambda_1(q) < \lambda_2(q) \leqslant \lambda_3(q) \leqslant \cdots$ be all eigenvalues of the following problem

$$-\Delta\phi + q(x)\phi = \lambda\phi, \qquad \phi|_{\partial\Omega} = 0,$$

where $q(x) \in C(\overline{\Omega})$. We know that $\lambda_1(q)$ is simple and $\lambda_1(q)$ is strictly increasing in the sense that $q_1 \leqslant q_2$ and $q_1 \not\equiv q_2$ implies $\lambda_1(q_1) < \lambda_1(q_2)$. For convenience, we denote $\lambda_i = \lambda_i(0)$. Moreover, we denote by Φ_1 the eigenfunction corresponding to λ_1 with normalization $\|\Phi_1\|_\infty = 1$ and positivity in Ω.

Define $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0,\ x \in \partial\Omega\}$. It is well known that for any $a > \lambda_1$,

$$-\Delta u = (a - u)u, \qquad u|_{\partial\Omega} = 0$$

has a unique positive solution which we denote by θ_a. It is also known that the mapping $a \to \theta_a$ is strictly increasing, continuously differentiable from (λ_1, ∞) to $C^2(\Omega) \cap C_0(\overline{\Omega})$ and that $\theta_a \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1$. Moreover, we have $0 < \theta_a < a$ in Ω. Therefore, (1.2) has semi-trivial solutions $(\theta_a, 0)$ if $a > \lambda_1$ and $(0, \theta_c)$ if $c > \lambda_1$.

Due to maximum principle, we give an a priori estimate.

Lemma 2.1. *Suppose that (u, v) is a positive solution of (1.2). Then (u, v) satisfies*

$$u < \theta_a < a \quad \text{and} \quad v < c + \frac{da}{1+ma}.$$

Furthermore, $v > \theta_c$ if $c > \lambda_1$.

We state the non-existence of positive solutions to (1.2).

Theorem 2.1. *If $a \leqslant \lambda_1$ or $c \leqslant \lambda_1 - d/m$, then there is no positive solution of (1.2).*

Next, we set up the fixed point index theory for later use. Let X be a real Banach space and W a closed convex set of X. W is called a total wedge if $\beta W \subset W$ for all $\beta > 0$ and $\overline{W - W} = X$. A wedge is said to be a cone if $W \cap (-W) = \{0\}$. For $y \in W$, define $W_y = \{x \in X\colon\ y + \gamma x \in W \text{ for some } \gamma > 0\}$ and $S_y = \{x \in \overline{W}_y\colon\ -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge containing W, y and $-y$, while S_y is a closed subspace of X containing y. Let $F : W \to W$ be a compact operator with a fixed point $y \in W$, and denote by L the Fréchet derivative of F at y. Then L maps $\overline{W}_y$ into itself. We say that L has property α on $\overline{W}_y$ if there exist $t \in (0, 1)$ and $w \in \overline{W}_y \backslash S_y$ such that $w - tLw \in S_y$.

For an open subset $U \subset W$, define $index_W(F, U) = index(F, U, W) = deg_W(I - F, U, 0)$, where I is the identity map. If y is an isolated fixed point of F, then the fixed point index of F at y in W is defined by $index_W(F, y) = index(F, U(y), W)$, where $U(y)$ is a small open neighborhood of y in W.

We state a general result of Dancer [15] on the fixed point index with respect to the positive cone W (see also [5]).

Theorem 2.2. *Assume that $I - L$ is invertible on X.*

(i) *If L has property α on $\overline{W}_y$, then $index_W(F, y) = 0$.*
(ii) *If L does not have property α on $\overline{W}_y$, then $index_W(F, y) = (-1)^\sigma$, where σ is the sum of algebraic multiplicities of the eigenvalues of L which are greater than* 1.

Now we introduce the following notations:

(i) $X := C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$;
(ii) $W := P \oplus P$, where $P = \{\varphi \in C_0(\overline{\Omega})\colon\ \varphi(x) \geqslant 0,\ x \in \overline{\Omega}\}$;
(iii) $\mathcal{D} := \{(u, v) \in X\colon\ u < a,\ v < c + \frac{da}{1+ma}\}$;
(iv) $\mathcal{D}' := (\operatorname{int} \mathcal{D}) \cap W$.

For any $t \in [0, 1]$, define a positive compact operator $\mathcal{A}_t : \mathcal{D}' \to W$ by

$$\mathcal{A}_t(u, v) = (-\Delta + M)^{-1} \begin{pmatrix} tu(a - u - \frac{bv}{1+mu+kv}) + Mu \\ tv(c - v + \frac{du}{1+mu+kv}) + Mv \end{pmatrix},$$

where $M = \max\{ad, b(c + ad)\}$. It follows from the standard elliptic regularity theory that $\mathcal{A}_t$ is a completely continuous operator. Observe that (1.2) has a positive solution in W if and only if $\mathcal{A} := \mathcal{A}_1$ has a positive fixed point in $\mathcal{D}'$. If $a, c > \lambda_1$, then $(0, 0)$, $(\theta_a, 0)$, $(0, \theta_c)$ are the only non-negative fixed points of $\mathcal{A}$ which are not positive. The corresponding indices in W can be calculated in the following lemmas.

Lemma 2.2. *Assume $a > \lambda_1$.*

(i) *$index_W(\mathcal{A}, \mathcal{D}') = 1$.*
(ii) *$index_W(\mathcal{A}, (0, 0)) = 0$.*
(iii) *If $c > \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$, then $index_W(\mathcal{A}, (\theta_a, 0)) = 0$.*
(iv) *If $c < \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$, then $index_W(\mathcal{A}, (\theta_a, 0)) = 1$.*

Lemma 2.3. *Assume $c > \lambda_1$.*

(i) *If $a > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$, then $index_W(\mathcal{A}, (0, \theta_c)) = 0$.*
(ii) *If $a < \lambda_1(\frac{b\theta_c}{1+k\theta_c})$, then $index_W(\mathcal{A}, (0, \theta_c)) = 1$.*

We state the existence of positive solutions to (1.2), which can be found in [13].

Theorem 2.3.

(i) *If* $a > \lambda_1$ *and* $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c \leqslant \lambda_1$, *then* (1.2) *has at least a positive solution.*
(ii) *If* $a > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$ *and* $c > \lambda_1$, *then* (1.2) *has at least a positive solution.*

Remark 1. The result in Section 2 of [13] does not include the case $c = \lambda_1$, but this case follows easily from Theorem 2 in [14].

3. The effect of mutual interference between predators

Since large k represents strong interference between predators on the predation, we want to know more information about positive solutions when k is large enough. Fortunately, we get a complete understanding of the existence, uniqueness and stability of positive solutions for large k. Due to different asymptotic behaviors of positive solutions when k goes to ∞, our discussion below can be divided into several different cases. In all these cases, we obtain uniqueness results. Moreover, we find that when $c \leqslant \lambda_1$ and a is close to λ_1, for any $k \geqslant 0$, (1.2) has at most one positive solution. That is, the uniqueness does not necessarily need k to be large at this moment. Furthermore, we establish a more general result when a is close to λ_1 and k is bounded. The main idea in these arguments is reducing the proof of the uniqueness to the proof of the fact that any positive solution is non-degenerate and linearly stable. Our work is motivated by the paper [9]. But, we must point out that the situation here is different from that in [9].

It should be noted that when $c > \lambda_1$, we have established the uniqueness for large k in [13]. But in the proof there, we always assume that c is bounded away from λ_1. Here we remove this assumption and consider two possibilities $c \geqslant \lambda_1 + \epsilon$ and $\lambda_1 < c < \lambda_1 + \epsilon$ for any $\epsilon > 0$ small. In particular, the discussion below includes the case that c is large.

We first investigate the asymptotic behavior of positive solutions for large k when c is bounded away from λ_1.

Lemma 3.1. *Assume* $a > \lambda_1$. *For any* ϵ, δ *small, there exists* $\bar{k} = \bar{k}(\epsilon, \delta) > 0$ *large such that if* $k \geqslant \bar{k}$ *and* $c \geqslant \lambda_1 + \epsilon$, *then*

$$\|u - \theta_a\|_{C^1} + \|v - \theta_c\|_{C^1} \leqslant \delta,$$

where (u, v) *is any positive solution* (*if exists*) *of* (1.2).

Proof. Suppose that the conclusion is not true. Then there exist $k_i \to \infty$, $c_i \to c \in [\lambda_1 + \epsilon, +\infty)$ and a positive solution (u_i, v_i) of (1.2) with $(c, k) = (c_i, k_i)$ such that (u_i, v_i) is bounded away from (θ_a, θ_c). Recalling Lemma 2.1, by the L^p estimate and the Sobolev embedding theorems, we may assume $u_i \to u$ and $v_i \to v \geqslant \theta_c > 0$ in C^1. Then $1/(1 + mu_i + k_i v_i) \to 0$ in any compact subset of Ω. Therefore (u, v) satisfies

$$\begin{cases} -\Delta u = u(a - u), & x \in \Omega, \\ -\Delta v = v(c - v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Since $a, c > \lambda_1$, we have $u = \theta_a$ and $v = \theta_c$. This contradicts the assumption. The proof is completed. □

From the standard perturbation argument, we give the following property of positive solutions (if exist) to (1.2). One can see (ii) of Lemma 3.1 in [13] for details.

Lemma 3.2. *Assume* $a > \lambda_1$. *For any* $\epsilon > 0$ *small and any constant* $M > \lambda_1$ *large, there exists* $\tilde{k} = \tilde{k}(\epsilon, M) > 0$ *large such that if* $\lambda_1 + \epsilon \leqslant c < M$ *and* $k \geqslant \tilde{k}$, *then any positive solution* (*if exists*) *of* (1.2) *is non-degenerate and linearly stable.*

We state the non-existence and the uniqueness of positive solutions obtained in [13].

Theorem 3.1. *Assume* $a > \lambda_1$. *For any* $\epsilon > 0$ *small and any constant* $M > \lambda_1$ *large, there exists* $\tilde{k} = \tilde{k}(\epsilon, M) > 0$ *large such that if* $\lambda_1 + \epsilon \leqslant c < M$ *and* $k \geqslant \tilde{k}$, *then* (1.2) *has no positive solution for* $\lambda_1 < a \leqslant \lambda_1(\frac{b\theta_c}{1+k\theta_c})$ *and has a unique positive solution for* $a > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$.

Remark 2. The non-existence result in [13] does not include the case $a = \lambda_1(\frac{b\theta_c}{1+k\theta_c})$. But, resorting to bifurcation theory, we can obtain the non-existence for $\lambda_1 < a \leqslant \lambda_1(\frac{b\theta_c}{1+k\theta_c})$. The proof is similar to that in [16], which one can refer to for detailed explanations.

Next, we consider the case that c is large. The following result gives the uniqueness of positive solutions to (1.2) for large c. By Lemma 2 in [17], we find that there is no positive solution for large c provided that $\lambda_1 < a < \lambda_1 + b/k$. Hence, the condition $a \geqslant \lambda_1 + b/k$ is necessary in the following theorem. At the same time, we point out that the result below is for any fixed constant $k > 0$.

Theorem 3.2. *Assume $a \geqslant \lambda_1 + b/k$. Then* (1.2) *has a unique positive solution for sufficiently large c and it is asymptotically stable.*

Proof. First we show that any positive solution of (1.2) is non-degenerate and linearly stable for large c. Argue indirectly. Suppose that we can find $c_i \to \infty$ such that (1.2) with $c = c_i$ has a positive solution (u_i, v_i) which is either degenerate or unstable. Then there exists $(\xi_i, \zeta_i) \not\equiv (0, 0)$ such that the following linearized problem

$$\begin{cases} -\Delta\xi_i - \left(a - 2u_i - \dfrac{bv_i(1+kv_i)}{(1+mu_i+kv_i)^2}\right)\xi_i + \dfrac{bu_i(1+mu_i)}{(1+mu_i+kv_i)^2}\zeta_i = \mu_i\xi_i, & x \in \Omega, \\ -\Delta\zeta_i - \left(c_i - 2v_i + \dfrac{du_i(1+mu_i)}{(1+mu_i+kv_i)^2}\right)\zeta_i - \dfrac{dv_i(1+kv_i)}{(1+mu_i+kv_i)^2}\xi_i = \mu_i\zeta_i, & x \in \Omega, \\ \xi_i = \zeta_i = 0, & x \in \partial\Omega \end{cases} \tag{3.1}$$

has an eigenvalue μ_i with $Re(\mu_i) \leqslant 0$. We may assume $\|\xi_i\|_2^2 + \|\zeta_i\|_2^2 = 1$.

We claim $\|\zeta_i\|_2 \to 0$. By Kato's inequality,

$$\begin{aligned} -\Delta|\zeta_i| &\leqslant -Re\left(\frac{\bar{\zeta}_i}{|\zeta_i|}\Delta\zeta_i\right) \\ &\leqslant \left(c_i - 2v_i + \frac{du_i(1+mu_i)}{(1+mu_i+kv_i)^2}\right)|\zeta_i| + \frac{dv_i(1+kv_i)}{(1+mu_i+kv_i)^2}|\xi_i| + Re(\mu_i)|\zeta_i|. \end{aligned} \tag{3.2}$$

Multiplying (3.2) by $|\zeta_i|$ and integrating by parts, we obtain

$$\int_\Omega \big|\nabla|\zeta_i|\big|^2\,dx \leqslant \int_\Omega (c_i - 2v_i)|\zeta_i|^2\,dx + \frac{d}{m}\int_\Omega |\zeta_i|^2\,dx + \frac{d}{k}\int_\Omega |\xi_i||\zeta_i|\,dx.$$

By Hölder's inequality and the fact that $v_i > \theta_{c_i}$, we obtain

$$0 < \lambda_1(-c_i + 2\theta_{c_i})\int_\Omega |\zeta_i|^2\,dx < \int_\Omega \big|\nabla|\zeta_i|\big|^2\,dx + \int_\Omega (-c_i + 2v_i)|\zeta_i|^2\,dx \leqslant M \tag{3.3}$$

for some positive constant M. By virtue of Lemma 2.2 in [9], there exists $k_0 \in (1, 2^{2/3})$ such that

$$\lambda_1(-c_i + 2\theta_{c_i}) = -c_i + \lambda_1(2\theta_{c_i}) \geqslant (k_0 - 1)c_i.$$

Hence $\lambda_1(-c_i + 2\theta_{c_i}) \to \infty$ as $c_i \to \infty$. Together with (3.3), we obtain $\|\zeta_i\|_2 \to 0$.

Multiplying the first equality in (3.1) by $\bar{\xi}_i$ and integrating by parts, we have

$$\int_\Omega |\nabla\xi_i|^2\,dx = \int_\Omega \left(a - 2u_i - \frac{bv_i(1+kv_i)}{(1+mu_i+kv_i)^2}\right)|\xi_i|^2\,dx - b\int_\Omega \frac{u_i(1+mu_i)\zeta_i\bar{\xi}_i}{(1+mu_i+kv_i)^2}\,dx + \mu_i\int_\Omega |\xi_i|^2\,dx.$$

Multiplying the second equality in (3.1) by $\bar{\zeta}_i$ and integrating by parts, we have

$$\int_\Omega |\nabla\zeta_i|^2\,dx = \int_\Omega \left(c_i - 2v_i + \frac{du_i(1+mu_i)}{(1+mu_i+kv_i)^2}\right)|\zeta_i|^2\,dx + d\int_\Omega \frac{v_i(1+kv_i)\xi_i\bar{\zeta}_i}{(1+mu_i+kv_i)^2}\,dx + \mu_i\int_\Omega |\zeta_i|^2\,dx.$$

Adding the above two equalities, we obtain

$$\begin{aligned} \mu_i = &\int_\Omega |\nabla\xi_i|^2\,dx - \int_\Omega \left(a - 2u_i - \frac{bv_i(1+kv_i)}{(1+mu_i+kv_i)^2}\right)|\xi_i|^2\,dx + b\int_\Omega \frac{u_i(1+mu_i)\zeta_i\bar{\xi}_i}{(1+mu_i+kv_i)^2}\,dx \\ &+ \int_\Omega |\nabla\zeta_i|^2\,dx - \int_\Omega \left(c_i - 2v_i + \frac{du_i(1+mu_i)}{(1+mu_i+kv_i)^2}\right)|\zeta_i|^2\,dx - d\int_\Omega \frac{v_i(1+kv_i)\xi_i\bar{\zeta}_i}{(1+mu_i+kv_i)^2}\,dx, \end{aligned}$$

where $\bar{\xi}_i$ and $\bar{\zeta}_i$ are the complex conjugates of ξ_i and ζ_i. By Lemma 2.1, we know that $0 < u_i < \theta_a$ and $\theta_{c_i} < v_i < c_i + da$. It is easy to see that $Im(\mu_i)$ is bounded. On the other hand, since c_i is unbounded, we need (3.3) and the fact that $\int_\Omega |\nabla|\zeta_i||^2\,dx \leqslant \int_\Omega |\nabla\zeta_i|^2\,dx$ to show that $Re(\mu_i)$ is bounded from below. Thus μ_i is bounded as we assume $Re(\mu_i) \leqslant 0$. Since $\theta_{c_i} < v_i < \theta_{c_i+d/m}$ and $\theta_{c_i}/c_i \to 1$ in any compact subset of Ω as $i \to \infty$, then $u_i \to \theta_{a-b/k}$ in C^1. By the L^p estimate, we have $\|\xi_i\|_{W^{2,2}}$ is bounded. Hence we may assume that $\mu_i \to \mu$ with $Re(\mu) \leqslant 0$ and $\xi_i \to \xi \not\equiv 0$ in H_0^1 strongly. By letting $i \to \infty$ in (3.1), we see that ξ satisfies the following equation weakly (then strongly)

$$-\Delta\xi - (a - 2\theta_{a-b/k} - b/k)\xi = \mu\xi, \qquad \xi|_{\partial\Omega} = 0.$$

The self-adjointness of the above problem gives $\mu \in R$. Moreover, since $\|\xi\|_2 = 1$ and $a - b/k \geqslant \lambda_1$, we must have $a - b/k = \lambda_1$, $\mu = 0$ and $\xi = \beta\Phi_1/\|\Phi_1\|_2$ with $|\beta| = 1$. Again by Kato's inequality,

$$\begin{aligned} -\Delta|\xi_i| &\leqslant -Re\left(\frac{\bar{\xi}_i}{|\xi_i|}\Delta\xi_i\right) \\ &\leqslant \left(a - 2u_i - \frac{bv_i(1+kv_i)}{(1+mu_i+kv_i)^2}\right)|\xi_i| + \frac{bu_i(1+mu_i)}{(1+mu_i+kv_i)^2}|\zeta_i| + Re(\mu_i)|\xi_i|. \end{aligned}$$

Multiplying the above inequality by u_i, integrating by parts and using the equation for u_i, we obtain

$$\int_\Omega u_i^2|\xi_i|\,dx \leqslant \int_\Omega \frac{bmu_i^2 v_i}{(1+mu_i+kv_i)^2}|\xi_i|\,dx + \frac{bu_i^2(1+mu_i)}{(1+mu_i+kv_i)^2}|\zeta_i|. \tag{3.4}$$

Set $\hat{u}_i = u_i/\|u_i\|_\infty$. Then $\hat{u}_i$ satisfies

$$-\Delta\hat{u}_i = \left(a - u_i - \frac{bv_i}{1+mu_i+kv_i}\right)\hat{u}_i, \qquad \|\hat{u}_i\|_\infty = 1, \qquad \hat{u}_i|_{\partial\Omega} = 0.$$

By standard elliptic regularity theory, we may assume $\hat{u}_i \to \hat{u} > 0$ in C^1 and $\hat{u}$ satisfies

$$-\Delta\hat{u} = (a - \theta_{a-b/k} - b/k)\hat{u}, \qquad \hat{u}|_{\partial\Omega} = 0.$$

Since $a - b/k = \lambda_1$, we have $\hat{u} = \Phi_1$. Dividing both sides of (3.4) by $\|u_i\|_\infty^2$, we find that

$$\int_\Omega \left(\frac{u_i}{\|u_i\|_\infty}\right)^2|\xi_i|\,dx \leqslant \int_\Omega \frac{bm\hat{u}_i^2 v_i}{(1+mu_i+kv_i)^2}|\xi_i|\,dx + \frac{b\hat{u}_i^2(1+mu_i)}{(1+mu_i+kv_i)^2}|\zeta_i|. \tag{3.5}$$

Since $v_i \to \infty$ uniformly in any compact subset of Ω as $i \to \infty$, the right hand side of (3.5) converges to 0. However, the left hand side of (3.5) goes to $\int_\Omega \frac{\Phi_1^3}{\|\Phi_1\|_2}\,dx$, which is a contradiction. This shows that if c is sufficiently large, then any positive solution of (1.2) is non-degenerate and linearly stable. By compactness, $\mathcal{A}$ has at most finitely many positive fixed points in the region $\mathcal{D}'$. Let us denote them by (u_i, v_i) for $i = 1, 2, \ldots, l$. Due to the non-degeneracy and stability of any positive solution, $I - L$ is invertible in X and L has no eigenvalue greater than one, where L is the Fréchet derivative of $\mathcal{A}$ at (u_i, v_i). Hence L does not have the property α. By Theorem 2.2, we obtain $index_W(\mathcal{A}, (u_i, v_i)) = 1$. Since $a \geqslant \lambda_1 + \frac{b}{k} > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$, by Lemmas 2.2, 2.3 and the additivity property of the index,

$$\begin{aligned} 1 &= index_W(\mathcal{A}, \mathcal{D}') = index_W(\mathcal{A}, (0,0)) + index_W(\mathcal{A}, (\theta_a, 0)) + index_W(\mathcal{A}, (0, \theta_c)) + \sum_{i=1}^{l} index_W(\mathcal{A}, (u_i, v_i)) \\ &= 0 + 0 + 0 + l = l. \end{aligned}$$

Hence (1.2) has a unique positive solution for sufficiently large c. The proof is completed. □

If c is near λ_1, then any positive solution of (1.2) is close to $(\theta_a, 0)$. That is, Lemma 3.1 does not hold true any more. But in this case, we can still show that any positive solution of (1.2) is non-degenerate and linearly stable for large k.

Theorem 3.3. *Assume $a > \lambda_1$. For any $\epsilon > 0$ small, if $\lambda_1 < c < \lambda_1 + \epsilon$ and k is large, then the conclusions in Theorem* 3.1 *still hold true.*

Proof. We claim that any positive solution is non-degenerate and linearly stable in this case. Argue by contradiction again. Suppose that we can find $c_i \to \lambda_1+$, $k_i \to \infty$ such that (1.2) with $(c,k) = (c_i, k_i)$ has a positive solution (u_i, v_i) which is either degenerate or unstable. Then there exists $(\xi_i, \zeta_i) \not\equiv (0,0)$ such that the following linearized problem

$$\begin{cases} -\Delta\xi_i - \left(a - 2u_i - \frac{bv_i(1+k_iv_i)}{(1+mu_i+k_iv_i)^2}\right)\xi_i + \frac{bu_i(1+mu_i)}{(1+mu_i+k_iv_i)^2}\zeta_i = \mu_i\xi_i, & x \in \Omega, \\ -\Delta\zeta_i - \left(c_i - 2v_i + \frac{du_i(1+mu_i)}{(1+mu_i+k_iv_i)^2}\right)\zeta_i - \frac{dv_i(1+k_iv_i)}{(1+mu_i+k_iv_i)^2}\xi_i = \mu_i\zeta_i, & x \in \Omega, \\ \xi_i = \zeta_i = 0, & x \in \partial\Omega \end{cases} \tag{3.6}$$

has an eigenvalue μ_i with $Re(\mu_i) \leqslant 0$. We may assume $\|\xi_i\|_2^2 + \|\zeta_i\|_2^2 = 1$.

By the L^p estimate and the Sobolev embedding theorems, we may assume $u_i \to u$ and $v_i \to v$ in C^1. We first show that if $c_i \to \lambda_1+$ and $k_i \to \infty$, then $(u_i, v_i) \to (u, v) = (\theta_a, 0)$. Set $\tilde{v}_i = v_i/\|v_i\|_\infty$. From the equation of v_i, it follows that $\tilde{v}_i$ satisfies

$$-\Delta \tilde{v}_i = \left(c_i - v_i + \frac{du_i}{1 + mu_i + k_i v_i}\right)\tilde{v}_i, \qquad \|\tilde{v}_i\|_\infty = 1, \qquad \tilde{v}_i|_{\partial\Omega} = 0. \tag{3.7}$$

Since $0 \leqslant 1/(1 + mu_i + k_i v_i) \leqslant 1$, we may assume $1/(1 + mu_i + k_i v_i) \rightharpoonup h$ with $0 \leqslant h \leqslant 1$ in L^2. By the L^p estimate and the Sobolev embedding theorems again, we have $\tilde{v}_i \to \tilde{v}$ in C^1. Passing to the limit in (3.7), we see $\tilde{v}$ satisfies the following equation weakly

$$-\Delta \tilde{v} = (\lambda_1 - v + duh)\tilde{v}, \qquad \|\tilde{v}\|_\infty = 1, \qquad \tilde{v}|_{\partial\Omega} = 0.$$

By Harnack's inequality, we obtain $\tilde{v} > 0$ in Ω. Recall the equation of v_i,

$$-\Delta v_i = \left(c_i - v_i + \frac{du_i}{1 + mu_i + k_i\|v_i\|_\infty \tilde{v}_i}\right)v_i, \qquad v_i|_{\partial\Omega} = 0. \tag{3.8}$$

If we can show $k_i\|v_i\|_\infty \to \infty$ as $i \to \infty$, then passing to the limit in (3.8), we get

$$-\Delta v = (\lambda_1 - v)v, \qquad v|_{\partial\Omega} = 0.$$

So $v_i \to v \equiv 0$, $\tilde{v}_i \to \tilde{v} = \Phi_1$. From the equation of u_i, we obtain $u_i \to \theta_a$ easily. Hence, it suffices to show that $k_i\|v_i\|_\infty \to \infty$ as $i \to \infty$. If this is not true, we may assume $k_i\|v_i\|_\infty$ is uniformly bounded. Set $\chi_i = k_i v_i$. Then χ_i satisfies

$$-\Delta \chi_i = \left(c_i - v_i + \frac{du_i}{1 + mu_i + \chi_i}\right)\chi_i, \qquad \chi_i|_{\partial\Omega} = 0. \tag{3.9}$$

Since $\|\chi_i\|_\infty = k_i\|v_i\|_\infty$ is bounded, by the L^p estimate and the Sobolev embedding theorems, we may assume $\chi_i \to \chi \geqslant 0$ and χ satisfies

$$-\Delta \chi = \left(\lambda_1 + \frac{d\theta_a}{1 + m\theta_a + \chi}\right)\chi, \qquad \chi|_{\partial\Omega} = 0. \tag{3.10}$$

If $\chi \geqslant 0$, $\not\equiv 0$, by the maximum principle, we have $\chi > 0$. By (3.10), we see $\lambda_1(-\frac{d\theta_a}{1+m\theta_a+\chi}) = \lambda_1$, which is a contradiction. If $\chi \equiv 0$, then considering the equation for $\tilde{\chi}_i = \chi_i/\|\chi_i\|_\infty$, we can get a contradiction similarly. Thus $k_i\|v_i\|_\infty \to \infty$ and then $u_i \to \theta_a$, $v_i \to 0$, $\tilde{v}_i = v_i/\|v_i\|_\infty \to \tilde{v} = \Phi_1$ as $i \to \infty$ in C^1.

Similar to the proof of Theorem 3.2, we can assume that $\mu_i \to \mu$ with $Re(\mu) \leqslant 0$ and $(\xi_i, \zeta_i) \to (\xi, \zeta)$ in H_0^1 strongly, where $(\xi, \zeta) \not\equiv (0, 0)$. Letting $i \to \infty$ in (3.6), we see that ξ and ζ satisfy the following two single equations

$$\begin{cases} -\Delta\xi - (a - 2\theta_a)\xi = \mu\xi, & x \in \Omega, \\ -\Delta\zeta - \lambda_1\zeta = \mu\zeta, & x \in \Omega, \\ \xi = \zeta = 0, & x \in \partial\Omega. \end{cases}$$

Obviously $\mu \in R$. If $\xi \not\equiv 0$, then $\mu \geqslant \mu_1 = \lambda_1(-a + 2\theta_a) > \lambda_1(-a + \theta_a) = 0$, which contradicts our assumption. Hence $\xi_i \to \xi \equiv 0$, $\mu_i \to \mu = 0$ and $\zeta_i \to \alpha\Phi_1/\|\Phi_1\|_2$ with $|\alpha| = 1$. By Kato's inequality,

$$\begin{aligned} -\Delta|\zeta_i| &\leqslant -Re\left(\frac{\bar{\zeta}_i}{|\zeta_i|}\Delta\zeta_i\right) \\ &\leqslant \frac{dv_i(1 + k_i v_i)}{(1 + mu_i + k_i v_i)^2}|\xi_i| + \left[c_i - 2v_i + \frac{du_i(1 + mu_i)}{(1 + mu_i + k_i v_i)^2}\right]|\zeta_i| + Re(\mu_i)|\zeta_i|. \end{aligned}$$

Multiplying the above inequality by v_i and integrating over Ω, after a simple rearrangement we obtain

$$\int_\Omega v_i^2|\zeta_i|\,dx \leqslant \int_\Omega \frac{dv_i^2(1 + k_i v_i)}{(1 + mu_i + k_i v_i)^2}|\xi_i|\,dx.$$

Dividing both sides of the above inequality by $\|v_i\|_\infty^2$, we obtain

$$\int_\Omega \left(\frac{v_i}{\|v_i\|_\infty}\right)^2|\zeta_i|\,dx \leqslant d\int_\Omega \left(\frac{v_i}{\|v_i\|_\infty}\right)^2|\xi_i|\,dx. \tag{3.11}$$

Since $\|\xi_i\|_2 \to 0$, the right hand side of (3.11) converges to 0 by Hölder's inequality. However, the above discussion tells us that $\int_\Omega(\frac{v_i}{\|v_i\|_\infty})^2|\zeta_i|\,dx \to \int_\Omega \frac{\Phi_1^3}{\|\Phi_1\|_2}\,dx$ as $i \to \infty$. This contradiction completes the proof of our claim.

By compactness, $\mathcal{A}$ has at most finitely many positive fixed points in the region $\mathcal{D}'$. Let us denote them by (u_i, v_i) for $i = 1, 2, \ldots, l$. Similar to the proof in [9], it follows $index_W(\mathcal{A}, (u_i, v_i)) = 1$ from the non-degeneracy and stability of any positive solution. By the additivity property of the index, we see that the conclusions in Theorem 3.1 still hold true. □

In what follows, we concentrate on the case $c \leqslant \lambda_1$. The existence problem for this case is completely understood. One can see Lemma 2.1 and (i) of Theorem 2.3. Our aim is to better understand the number and the stability of positive solutions when $c \leqslant \lambda_1$, $a > \lambda_1$ and k is sufficiently large. We first consider the asymptotic behavior of positive solutions when k goes to ∞ and c $(< \lambda_1)$ is bounded away from λ_1. It turns out that if k is sufficiently large, (1.2) has only one type of positive solutions. More precisely, if $k \to \infty$, and if (u, v) is any positive solution of (1.2), then (u, kv) is close to (θ_a, χ), where χ is a positive solution of the following equation

$$-\Delta\chi = \left(c + \frac{d\theta_a}{1 + m\theta_a + \chi}\right)\chi, \qquad \chi|_{\partial\Omega} = 0. \tag{3.12}$$

We first study the existence of positive solutions to (3.12).

Lemma 3.3. *Assume $a > \lambda_1$. Then* (3.12) *has a positive solution if and only if $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c < \lambda_1$. Moreover, any positive solution of* (3.12) *is non-degenerate and linearly stable.*

Proof. If (3.12) has a positive solution, then $c = \lambda_1(-\frac{d\theta_a}{1+m\theta_a+\chi}) < \lambda_1$. On the other hand, from the monotonicity of principal eigenvalues, it follows that $c > \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$ easily.

Next, we show that (3.12) has at least a positive solution for $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c < \lambda_1$. To this end, we first prove that for any $\epsilon > 0$ small, there exists a constant $C = C(\epsilon) > 0$ such that $\|\chi\|_\infty < C$ for any positive solution of (3.12) with $c \leqslant \lambda_1 - \epsilon$. Suppose this is not true. Then we may assume that there exist $\epsilon_0 > 0$, $c_i \to c \leqslant \lambda_1 - \epsilon_0$, χ_i positive solutions of (3.12) with $c = c_i$ such that $\|\chi_i\|_\infty \to \infty$. Set $\tilde{\chi}_i = \chi_i/\|\chi_i\|_\infty$. Then $\tilde{\chi}_i$ satisfies

$$-\Delta\tilde{\chi}_i = \left(c_i + \frac{d\theta_a}{1 + m\theta_a + \|\chi_i\|_\infty\tilde{\chi}_i}\right)\tilde{\chi}_i, \qquad \|\tilde{\chi}_i\|_\infty = 1, \qquad \tilde{\chi}_i|_{\partial\Omega} = 0.$$

Since $0 \leqslant 1/(1 + m\theta_a + \|\chi_i\|_\infty\tilde{\chi}_i) \leqslant 1$, we may assume $1/(1 + m\theta_a + \|\chi_i\|_\infty\tilde{\chi}_i) \rightharpoonup h$ with $0 \leqslant h \leqslant 1$ in L^2. By the L^p estimate and the Sobolev embedding theorems, we have $\tilde{\chi}_i \to \tilde{\chi} \geqslant 0, \not\equiv 0$ in C^1 and $\tilde{\chi}$ satisfies

$$-\Delta\tilde{\chi} = (c + d\theta_a h)\tilde{\chi}, \qquad \tilde{\chi}|_{\partial\Omega} = 0.$$

Since $0 \leqslant h \leqslant 1$, the Harnack inequality is applicable, and we obtain $\tilde{\chi} > 0$ in Ω. From $\|\chi_i\|_\infty \to \infty$, it follows that $h \equiv 0$. Then we have $-\Delta\tilde{\chi} = c\tilde{\chi}$, which implies $c = \lambda_1$. This is a contradiction. Hence we have established the desired a priori estimate.

Set $\tilde{\mathcal{D}} = \{\chi \in P\colon \|\chi\|_\infty < C(\epsilon) + 1\}$. Define $\mathcal{B}_\tau : \tilde{\mathcal{D}} \to P$ by

$$\mathcal{B}_\tau\chi = (-\Delta + M)^{-1}\left(\left(\tau + \frac{d\theta_a}{1 + m\theta_a + \chi}\right)\chi + M\chi\right),$$

where M is a constant satisfying $d/m < M$. Observe that (3.12) has a positive solution if and only if $\mathcal{B}_c$ has a positive fixed point in $\tilde{\mathcal{D}}$. By virtue of our a priori estimate and the homotopic invariance property of the index, we obtain $index_P(\mathcal{B}_\tau, \tilde{\mathcal{D}}) \equiv constant$ for all $\tau \leqslant \lambda_1 - \epsilon$. If $\tau < \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$, then $\chi = 0$ is the only non-negative fixed point of $\mathcal{B}_\tau$ in $\tilde{\mathcal{D}}$. Hence for $\tau < \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$, $index_P(\mathcal{B}_\tau, \tilde{\mathcal{D}}) = index_P(\mathcal{B}_\tau, 0)$. By some simple calculations, we obtain that

$$index_P(\mathcal{B}_\tau, 0) = \begin{cases} 1, & \tau < \lambda_1(-\frac{d\theta_a}{1+m\theta_a}); \\ 0, & \tau > \lambda_1(-\frac{d\theta_a}{1+m\theta_a}). \end{cases}$$

Then for $\tau < \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$, we have $index_P(\mathcal{B}_\tau, \tilde{\mathcal{D}}) = 1$.

Suppose that there is no positive solution to (3.12) for $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c < \lambda_1$. Then $index_P(\mathcal{B}_\tau, \tilde{\mathcal{D}}) = index_P(\mathcal{B}_\tau, 0) = 0$ for $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < \tau \leqslant \lambda_1 - \epsilon$. The homotopic invariance property of the index tells us that there is a contradiction. Hence $\mathcal{B}_\tau$ has at least a positive fixed point in $\tilde{\mathcal{D}}$ for $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < \tau \leqslant \lambda_1 - \epsilon$. Namely, (3.12) has at least a positive solution if $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c < \lambda_1$.

It remains to prove the stability of any positive solution χ_0 of (3.12). To this end, consider the eigenvalue problem

$$-\Delta\phi - \left(c + \frac{d\theta_a(1 + m\theta_a)}{(1 + m\theta_a + \chi_0)^2}\right)\phi = \mu\phi, \qquad \phi|_{\partial\Omega} = 0.$$

Clearly, $\mu_1(-\frac{d\theta_a(1+m\theta_a)}{(1+m\theta_a+\chi_0)^2}-c) > \lambda_1(-\frac{d\theta_a}{1+m\theta_a+\chi_0}-c)=0$. Hence, any positive solution of (3.12) is non-degenerate and linearly stable. The proof is completed. □

Remark 3. Since any positive solution of (3.12) is non-degenerate and linearly stable, then (3.12) has at most a positive solution. Thus (3.12) has a unique positive solution when $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c < \lambda_1$.

As noted before, the next theorem shows rigorously that all the positive solutions to (1.2) are of only one type when $c \leqslant \lambda_1 - \epsilon$ and k is large.

Lemma 3.4. *Assume $a > \lambda_1$. Let (u, v) be any positive solution of* (1.2)*. For any $\epsilon, \delta > 0$ small, there exists $\tilde{k} = k(\epsilon, \delta)$ large such that if $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c \leqslant \lambda_1 - \epsilon$ and $k > \tilde{k}$, then $\|u-\theta_a\|_{C^1} + \|kv-\chi\|_{C^1} \leqslant \delta$, where χ is a positive solution of* (3.12)*.*

Proof. Argue by contradiction. Suppose we can find $\epsilon_0, \delta_0 > 0$, $c_i \to c \in [\lambda_1(-\frac{d\theta_a}{1+m\theta_a}), \lambda_1 - \epsilon_0]$, $k_i \to \infty$ and a positive solution (u_i, v_i) of (1.2) with $(c, k) = (c_i, k_i)$ such that

$$\|u_i-\theta_a\|_{C^1} + \|k_i v_i - \chi\|_{C^1} \geqslant \delta_0.$$

From Lemma 2.1, we have $0 < u_i < \theta_a$, $\theta_{c_i} < v_i < c_i + d/m$. By the L^p estimate and the Sobolev embedding theorems, we may assume the existence of a subsequence (if necessary) such that $u_i \to u$ and $v_i \to v$ in C^1. In what follows, we first show $k_i\|v_i\|_\infty$ is uniformly bounded, which implies $v_i \to v = 0$. Argue indirectly. Suppose $k_i\|v_i\|_\infty \to \infty$ as $i \to \infty$. Similar to the proof of Theorem 3.3, we obtain $\tilde{v}_i = v_i/\|v_i\|_\infty \to \tilde{v} > 0$. Passing to the limit in the following equation

$$-\Delta v_i = \left(c_i - v_i + \frac{du_i}{1+mu_i+k_i\|v_i\|_\infty \tilde{v}_i}\right)v_i, \qquad v_i|_{\partial\Omega} = 0,$$

we have $-\Delta v = (c-v)v$. Since $c < \lambda_1$, then $v = 0$. Thus the limit of $\tilde{v}_i$ satisfies $-\Delta\tilde{v} = c\tilde{v}$. From $\tilde{v} > 0$, it follows $c = \lambda_1$, which is a contradiction. Therefore, $k_i\|v_i\|_\infty$ is uniformly bounded, which implies $v_i \to v = 0$, $u_i \to u = \theta_a$. Set $\chi_i = k_i v_i$. Then $\|\chi_i\|_\infty$ is bounded and χ_i satisfies (3.9). By the L^p estimate and the Sobolev embedding theorems, we may assume $\chi_i \to \chi$ in C^1 and χ satisfies

$$-\Delta\chi = \left(c + \frac{d\theta_a}{1+m\theta_a+\chi}\right)\chi, \qquad \chi|_{\partial\Omega} = 0.$$

If $\chi \geqslant 0, \not\equiv 0$, then by the maximum principle, we have $\chi > 0$, which completes the proof. If $\chi \equiv 0$, then let $\tilde{\chi}_i = \chi_i/\|\chi_i\|_\infty$ and $\tilde{\chi}_i$ satisfies

$$-\Delta\tilde{\chi}_i = \left(c_i - v_i + \frac{du_i}{1+mu_i+\chi_i}\right)\tilde{\chi}_i, \qquad \|\tilde{\chi}_i\|_\infty = 1, \qquad \tilde{\chi}_i|_{\partial\Omega} = 0.$$

Passing to the limit, we obtain $\tilde{\chi}_i \to \tilde{\chi} > 0$ and $\tilde{\chi}$ satisfies

$$-\Delta\tilde{\chi} = \left(c + \frac{d\theta_a}{1+m\theta_a}\right)\tilde{\chi}, \qquad \tilde{\chi}|_{\partial\Omega} = 0,$$

which implies $c = \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$. On the other hand, we know that (3.12) has a positive solution branch bifurcating from $(c; \chi) = (\lambda_1(-\frac{d\theta_a}{1+m\theta_a}); 0)$. Hence, we can find $c = \bar{c}_i \to \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$ such that (3.12) with $c = \bar{c}_i$ has a positive solution $\bar{\chi}_i$ converging to 0 in L^∞. Thus $(c_i; k_i v_i)$ is close to $(\bar{c}_i; \bar{\chi}_i)$ for sufficiently large i. This again contradicts our assumption. The proof is completed. □

Theorem 3.4. *Assume $a > \lambda_1$. For any $\epsilon > 0$ small, if $\lambda_1(-\frac{d\theta_a}{1+m\theta_a}) < c \leqslant \lambda_1 - \epsilon$ and k is sufficiently large, then* (1.2) *has a unique positive solution. Moreover, it is non-degenerate and linearly stable.*

Proof. By (i) of Theorem 2.3, we only need to prove the uniqueness. We first claim that any positive solution of (1.2) is non-degenerate and linearly stable. In fact, it suffices to show that the corresponding linearized eigenvalue problem has no eigenvalue μ with $Re(\mu) \leqslant 0$. To do this, a contradiction argument will be used again by assuming that (1.2) has a positive solution (u_i, v_i) which is either degenerate or linearly unstable for sequences $k_i \to \infty$ and $c_i \to c \leqslant \lambda_1 - \epsilon$. Thus there exist μ_i with $Re(\mu_i) \leqslant 0$ and $(\xi_i, \zeta_i) \not\equiv (0, 0)$ such that (3.6) holds. From Lemma 3.4, it follows that $u_i \to \theta_a$, $v_i \to 0$, $k_i v_i \to \chi$, where χ is a positive solution of (3.12). Similar to the proof of Theorem 3.2, we can assume $\mu_i \to \mu$ with $Re(\mu) \leqslant 0$ and $(\xi_i, \zeta_i) \to (\xi, \zeta) \not\equiv (0, 0)$ in H_0^1 strongly. Letting $i \to \infty$ in (3.6), we see that (ξ, ζ) satisfies

$$\begin{cases} -\Delta\xi - (a-2\theta_a)\xi + \dfrac{b\theta_a(1+m\theta_a)}{(1+m\theta_a+\chi)^2}\zeta = \mu\xi, & x\in\Omega,\\ -\Delta\zeta - \left(c+\dfrac{d\theta_a(1+m\theta_a)}{(1+m\theta_a+\chi)^2}\right)\zeta = \mu\zeta, & x\in\Omega,\\ \xi=\zeta=0, & x\in\partial\Omega. \end{cases}$$

Obviously, $\mu\in R$. If $\zeta\equiv 0$, then we have $\mu>0$, which is impossible. Hence $\zeta\not\equiv 0$. Then

$$\mu\geqslant\mu_1=\lambda_1\left(-\frac{d\theta_a(1+m\theta_a)}{(1+m\theta_a+\chi)^2}-c\right)>\lambda_1\left(-\frac{d\theta_a}{1+m\theta_a+\chi}-c\right)=0,$$

a contradiction again. Therefore, any positive solution of (1.2) is non-degenerate and linearly stable.

By compactness arguments, $\mathcal{A}$ has at most finitely many positive fixed points in the region $\mathcal{D}'$. Let us denote them by (u_i,v_i) for $i=1,2,\dots,l$ again. Similar to the proof in [9], it follows $index_W(\mathcal{A},(u_i,v_i))=1$ from the above discussion. By the additivity property of the index, we have

$$\begin{aligned}1&=index_W\big(\mathcal{A},\mathcal{D}'\big)=index_W\big(\mathcal{A},(0,0)\big)+index_W\big(\mathcal{A},(\theta_a,0)\big)+\sum_{i=1}^{l}index_W\big(\mathcal{A},(u_i,v_i)\big)\\&=0+0+l=l.\end{aligned}$$

The uniqueness follows. □

Remark 4. Assume $a>\lambda_1$. In order to get the uniqueness for any $c\in R$, it remains to show that when $\lambda_1-\epsilon<c\leqslant\lambda_1$ and k is sufficiently large, (1.2) has a unique positive solution. Similar to the proof of Theorem 3.3, we can still obtain that any positive solution of (1.2) is non-degenerate and linearly stable for $\lambda_1-\epsilon<c\leqslant\lambda_1$. The uniqueness follows similarly.

The above arguments tell us that for any fixed $a>\lambda_1$, if k is sufficiently large, then (1.2) has at most one positive solution. It should be noted that our uniqueness result is for any $c\in R$. In previous discussions, we always assume a is bounded away from λ_1. If we remove this assumption and suppose a is close to λ_1, we shall find that the uniqueness of positive solutions to (1.2) does not necessarily need k sufficiently large.

Theorem 3.5. *For any $\epsilon>0$ small, if $\lambda_1<a\leqslant\lambda_1+\epsilon$ and $c\leqslant\lambda_1$, then* (1.2) *has at most one positive solution for any $k\geqslant 0$, and it* (*if exists*) *is non-degenerate and linearly stable.*

Proof. We first show the non-degeneracy and stability of positive solutions. Argue by contradiction again. Suppose that we can find $a_i\to\lambda_1+$, $k_i\to k\geqslant 0$ and $c_i\leqslant\lambda_1$ such that (1.2) with $(a,c,k)=(a_i,c_i,k_i)$ has a positive solution (u_i,v_i) which is either degenerate or unstable. That is, there exist μ_i with $Re(\mu_i)\leqslant 0$ and $(\xi_i,\zeta_i)\not\equiv(0,0)$ such that (3.6) with $a=a_i$ holds. We may assume $\|\xi_i\|_2^2+\|\zeta_i\|_2^2=1$.

Since $0<u_i<\theta_{a_i}$, we have $u_i\to 0$ in L^∞. From the equation for v_i, we can deduce that

$$0\equiv\theta_{c_i}<v_i<\theta_{c_i+d\|\theta_{a_i}\|_\infty}. \tag{3.13}$$

From $v_i>0$, it follows $c_i+d\|\theta_{a_i}\|_\infty>\lambda_1$. Thus $\lambda_1-d\|\theta_{a_i}\|_\infty<c_i\leqslant\lambda_1$. This implies $c_i\to\lambda_1$, since $\|\theta_{a_i}\|_\infty\to 0$. By (3.13), we have $v_i\to 0$ in L^∞. Using these facts and the equations for u_i and v_i, one can easily show by a compactness argument that

$$u_i/\|u_i\|_\infty\to\Phi_1,\qquad v_i/\|v_i\|_\infty\to\Phi_1\quad\text{in } C^1.$$

Thus one can rewrite u_i and v_i in the form

$$u_i=s_i\cos\varpi_i(\Phi_1+\rho_i),\qquad v_i=s_i\sin\varpi_i(\Phi_1+z_i), \tag{3.14}$$

where $\rho_i,z_i\to 0$ in C^1, $(\rho_i,\Phi_1)_2=(z_i,\Phi_1)_2=0$, $\varpi_i\in(0,\pi/2)$, and

$$s_i=\big(\|u_i\|_\infty^2/\|\Phi_1+\rho_i\|_\infty^2+\|v_i\|_\infty^2/\|\Phi_1+z_i\|_\infty^2\big)^{1/2}.$$

Since $Re(\mu_i)\leqslant 0$, as in the proof of Theorem 3.2, it can be easily shown that μ_i is bounded. Thus ξ_i, ζ_i are bounded in $W^{2,2}$. Without loss of generality, we may assume that $\mu_i\to\mu$ with $Re(\mu)\leqslant 0$ and $(\xi_i,\zeta_i)\to(\xi,\zeta)\not\equiv(0,0)$ in H_0^1 strongly. Taking the limit in the linearized problem (3.6) with $a=a_i$, we obtain

$$\begin{cases}\Delta\xi+\lambda_1\xi+\mu\xi=0, & x\in\Omega,\\ \Delta\zeta+\lambda_1\zeta+\mu\zeta=0, & x\in\Omega,\\ \xi=\zeta=0, & x\in\partial\Omega.\end{cases}$$

Thus we must have $\mu = 0$, $\xi = h\Phi_1$ and $\zeta = k\Phi_1$, where h, k are some real numbers and $(h,k) \neq (0,0)$. By Kato's inequality, similar to the proof of Theorem 3.3, we find the inequality (3.11) still holds true. Passing to the limit in (3.11), we obtain

$$\int_\Omega \Phi_1^2 |\zeta| \,\mathrm{d}x \leqslant d \int_\Omega \Phi_1^2 |\xi| \,\mathrm{d}x.$$

Hence $h \neq 0$ and $|k| \leqslant d|h|$. Thus if we rescale (ξ_i, ζ_i) properly, we can assume that $\xi_i \to \Phi_1$, $\zeta_i \to p\Phi_1$, where $|p| \leqslant d$. By rescaling (ξ_i, ζ_i) suitably once more if necessary, we may assume that

$$\xi_i = \Phi_1 + \xi_i', \qquad (\Phi_1, \xi_i')_2 = 0, \quad \xi_i' \to 0,$$
$$\zeta_i = p_i(\Phi_1 + \zeta_i'), \qquad (\Phi_1, \zeta_i')_2 = 0, \quad \zeta_i' \to 0, \; p_i \to p.$$

Now multiplying the equation for ξ_i in (3.6) with $a = a_i$ by Φ_1, integrating over Ω and noting that the new expressions of ξ_i and ζ_i as above, we have

$$\lambda_1 \int_\Omega \Phi_1^2 \,\mathrm{d}x = a_i \int_\Omega \Phi_1^2 \,\mathrm{d}x - \int_\Omega \left[2u_i + \frac{bv_i(1+k_iv_i)}{(1+mu_i+k_iv_i)^2}\right]\xi_i\Phi_1 \,\mathrm{d}x$$
$$- b\int_\Omega \frac{u_i(1+mu_i)}{(1+mu_i+k_iv_i)^2}\zeta_i\Phi_1 \,\mathrm{d}x + \mu_i \int_\Omega \Phi_1^2 \,\mathrm{d}x. \tag{3.15}$$

Passing to a subsequence if needed, we may assume $k_is_i \to k^* \in [0,\infty]$ and $\varpi_i \to \varpi \in [0,\pi/2]$. Then by (3.14) and (3.15), we obtain

$$\lim_{i\to\infty}\left(\frac{a_i-\lambda_1}{s_i} + \frac{\mu_i}{s_i}\right)\int_\Omega \Phi_1^2 \,\mathrm{d}x = 2\cos\varpi \int_\Omega \Phi_1^3 \,\mathrm{d}x + bp\cos\varpi \int_\Omega \frac{\Phi_1^3}{(1+k^*\sin\varpi\,\Phi_1)^2}\,\mathrm{d}x$$
$$+ b\sin\varpi \int_\Omega \frac{\Phi_1^3}{1+k^*\sin\varpi\,\Phi_1}\,\mathrm{d}x. \tag{3.16}$$

Dividing the equation for u_i by $s_i\cos\varpi_i$, multiplying by Φ_1 and integrating by parts lead to

$$\lambda_1 \int_\Omega \Phi_1^2 \,\mathrm{d}x = a_i \int_\Omega \Phi_1^2 \,\mathrm{d}x - \int_\Omega (\Phi_1 + \rho_i)\left(u_i + \frac{bv_i}{1+mu_i+k_iv_i}\right)\Phi_1 \,\mathrm{d}x.$$

Hence

$$\lim_{i\to\infty}\frac{a_i-\lambda_1}{s_i}\int_\Omega \Phi_1^2 \,\mathrm{d}x = \cos\varpi \int_\Omega \Phi_1^3 \,\mathrm{d}x + b\sin\varpi \int_\Omega \frac{\Phi_1^3}{1+k^*\sin\varpi\,\Phi_1}\,\mathrm{d}x. \tag{3.17}$$

From (3.16) and (3.17), we have

$$\lim_{i\to\infty}\frac{\mu_i}{s_i}\int_\Omega \Phi_1^2 \,\mathrm{d}x = \cos\varpi \int_\Omega \Phi_1^3 \,\mathrm{d}x + bp\cos\varpi \int_\Omega \frac{\Phi_1^3}{(1+k^*\sin\varpi\,\Phi_1)^2}\,\mathrm{d}x. \tag{3.18}$$

Dividing the equation for v_i by $s_i\sin\varpi_i$, multiplying by Φ_1 and integrating by parts, we get

$$\lim_{i\to\infty}\frac{c_i-\lambda_1}{s_i}\int_\Omega \Phi_1^2 \,\mathrm{d}x = \sin\varpi \int_\Omega \Phi_1^3 \,\mathrm{d}x - d\cos\varpi \int_\Omega \frac{\Phi_1^3}{1+k^*\sin\varpi\,\Phi_1}\,\mathrm{d}x. \tag{3.19}$$

Since $c_i \leqslant \lambda_1$, by (3.19), we can deduce $\sin\varpi \leqslant d\cos\varpi$. Hence $\varpi \neq \pi/2$ and then $0 \leqslant \varpi < \pi/2$. If $p_i \to p = 0$, then

$$\lim_{i\to\infty}\frac{\mu_i}{s_i}\int_\Omega \Phi_1^2 \,\mathrm{d}x = \cos\varpi \int_\Omega \Phi_1^3 \,\mathrm{d}x > 0, \tag{3.20}$$

which contradicts our assumption that $Re(\mu_i) \leqslant 0$. Hence $p \neq 0$. Now we use

$$\int_\Omega \zeta_i v_i\left(c_i - v_i + \frac{du_i}{1+mu_i+k_iv_i}\right)\mathrm{d}x = \int_\Omega \zeta_i(-\Delta v_i)\,\mathrm{d}x = \int_\Omega v_i(-\Delta\zeta_i)\,\mathrm{d}x$$
$$= \int_\Omega \left[\left(c_i - 2v_i + \frac{du_i(1+mu_i)}{(1+mu_i+k_iv_i)^2}\right)\zeta_i + \frac{dv_i(1+k_iv_i)}{(1+mu_i+k_iv_i)^2}\xi_i + \mu_i\zeta_i\right]v_i \,\mathrm{d}x$$

to obtain

$$\int_\Omega \zeta_i v_i^2 \,dx \leqslant d \int_\Omega \frac{v_i^2(1+k_i v_i)}{(1+mu_i+k_i v_i)^2}\xi_i \,dx.$$

Dividing the above inequality by $p_i(s_i \sin\varpi_i)^2$, taking the real parts and passing to the limit, we have

$$\int_\Omega \Phi_1^3 \,dx \leqslant Re\left(\frac{1}{p}\right)\int_\Omega \frac{d\Phi_1^3}{1+k^*\sin\varpi\,\Phi_1}\,dx.$$

Hence $Re(p) > 0$. By (3.18), $Re(\mu_i) > 0$ for large i, which contradicts our assumption.

From the above arguments, we see that any positive solution of (1.2) is non-degenerate and linearly stable. Since the subsequent proof is similar to the proof of the uniqueness in Theorem 3.4, we omit the details. The proof is completed. $\square$

In fact, when a is close to λ_1 and k is bounded, we can know a little more than Theorem 3.5. In particular, a sufficient and necessary condition for the existence of positive solutions to (1.2) is derived.

Theorem 3.6. *For any $\epsilon > 0$ small, there exists $\hat{k} = \hat{k}(\epsilon) > 0$ such that if $a \in (\lambda_1, \lambda_1+\epsilon)$ and $k \in [0,\hat{k}]$, then (1.2) has no positive solution for $c \notin (c_0, c_1)$, and has a unique positive solution for $c \in (c_0, c_1)$. Moreover, the unique positive solution is asymptotically stable. Here $c_0 = \lambda_1(-\frac{d\theta_a}{1+m\theta_a})$ and c_1 is defined by $a = \lambda_1(\frac{b\theta_{c_1}}{1+k\theta_{c_1}})$.*

Proof. For any $\epsilon > 0$ small, let $\hat{k} = b/\epsilon$. We first claim that for $a \in (\lambda_1, \lambda_1+\epsilon)$, $k \in [0,\hat{k}]$ and any c, any positive solution (if exists) is non-degenerate and linearly stable. Argue indirectly again. Suppose there exist $a_i \to \lambda_1+$, $k_i \in [0,\hat{k}]$ and c_i such that (1.2) with $(a,c,k) = (a_i,c_i,k_i)$ has a positive solution (u_i,v_i) which is either degenerate or unstable. That is, there exist μ_i with $Re(\mu_i) \leqslant 0$ and $(\xi_i,\zeta_i) \not\equiv (0,0)$ such that (3.6) holds.

Since $0 < u_i < \theta_{a_i}$, then $u_i \to 0$ in L^∞. From the equation for v_i, we again deduce that $c_i > \lambda_1 - d\|\theta_{a_i}\|_\infty \to \lambda_1$. We may assume by choosing a subsequence that $c_i \to c^* \in [\lambda_1,\infty]$ and $k_i \to k \in [0,\hat{k}]$. Then we have $v_i \to \theta_{c^*}$. Here we understand $\theta_\infty = \infty$. Hence

$$u_i + \frac{bv_i}{1+mu_i+k_iv_i} \to \frac{b\theta_{c^*}}{1+k\theta_{c^*}}.$$

But it follows from $u_i > 0$ and the equation for u_i that

$$a_i = \lambda_1\left(u_i + \frac{bv_i}{1+mu_i+k_iv_i}\right).$$

Passing to the limit, we obtain $\lambda_1 = \lambda_1(\frac{b\theta_{c^*}}{1+k\theta_{c^*}})$. Hence we must have $c_i \to c^* = \lambda_1$, which implies $c_i \to \lambda_1$. By (3.13), we have $v_i \to 0$ in L^∞.

Now we see obviously that everything in the proof of Theorem 3.5 carries over to the present case. Hence we obtain that for $a \in (\lambda_1, \lambda_1+\epsilon)$ and $k \in [0,\hat{k}]$, any positive solution of (1.2) is non-degenerate and linearly stable. Here we remove the restriction $c \leqslant \lambda_1$. Since $\hat{k} = b/\epsilon$, it follows from $k \leqslant \hat{k}$ that $\lambda_1 < a < \lambda_1+\epsilon \leqslant \lambda_1 + b/k$. Combined with Theorem 5 in [17], we know there is at least a positive solution to (1.2) if $c \in (c_0,c_1)$. Since the subsequent proof of the uniqueness is similar to the proof of Theorem 3.4, we omit the details.

It remains to prove that there is no positive solution if $c \notin (c_0,c_1)$. Suppose that (u,v) is any positive solution of (1.2). Then from $v > 0$ and the equation for v, it follows that

$$c = \lambda_1\left(v - \frac{du}{1+mu+kv}\right) > \lambda_1\left(-\frac{d\theta_a}{1+m\theta_a}\right).$$

Hence (1.2) has no positive solution if $c \leqslant c_0$. In what follows, we shall show that (1.2) has no positive solution if $c \geqslant c_1$. We argue indirectly. Suppose that for some $a' \in (\lambda_1, \lambda_1+\epsilon)$ and $k' \in [0,\hat{k}]$, there exists $c' \geqslant c_1$ such that (1.2) with $(a,c,k) = (a',c',k')$ has a positive solution. Set

$$\hat{c} = \sup\bigl\{c'':\ (1.2) \text{ has a positive solution for } (a,c,k) = \bigl(a',c'',k'\bigr)\bigr\}.$$

Clearly, $\hat{c} \geqslant c' \geqslant c_1$. By Lemma 2 in [17], we have $\hat{c} < \infty$. There are only two possibilities:

(i) $\hat{c} = c_1$. In this case, we must have $\hat{c} = c' = c_1$. Hence (1.2) has a positive solution $(\hat{u},\hat{v})$ for $c = \hat{c}$. And $(\hat{u},\hat{v})$ must be a degenerate positive solution of (1.2). Otherwise we can apply the implicit function theorem to extend this solution of (1.2) to the right of $\hat{c}$, which contradicts the definition of $\hat{c}$. However, by our claim as shown in the above, $(\hat{u},\hat{v})$ is non-degenerate. Clearly, there is a contradiction.

(ii) $\hat{c} > c_1$ $(> \lambda_1)$. By the definition of $\hat{c}$ and a simple compactness argument, (1.2) has a non-negative solution $(\hat{u},\hat{v})$ at $c = \hat{c}$. By virtue of the continuity, we have $\hat{v} \geqslant \theta_{\hat{c}}$. If $(\hat{u},\hat{v})$ is a positive solution of (1.2), then we get a contradiction in the

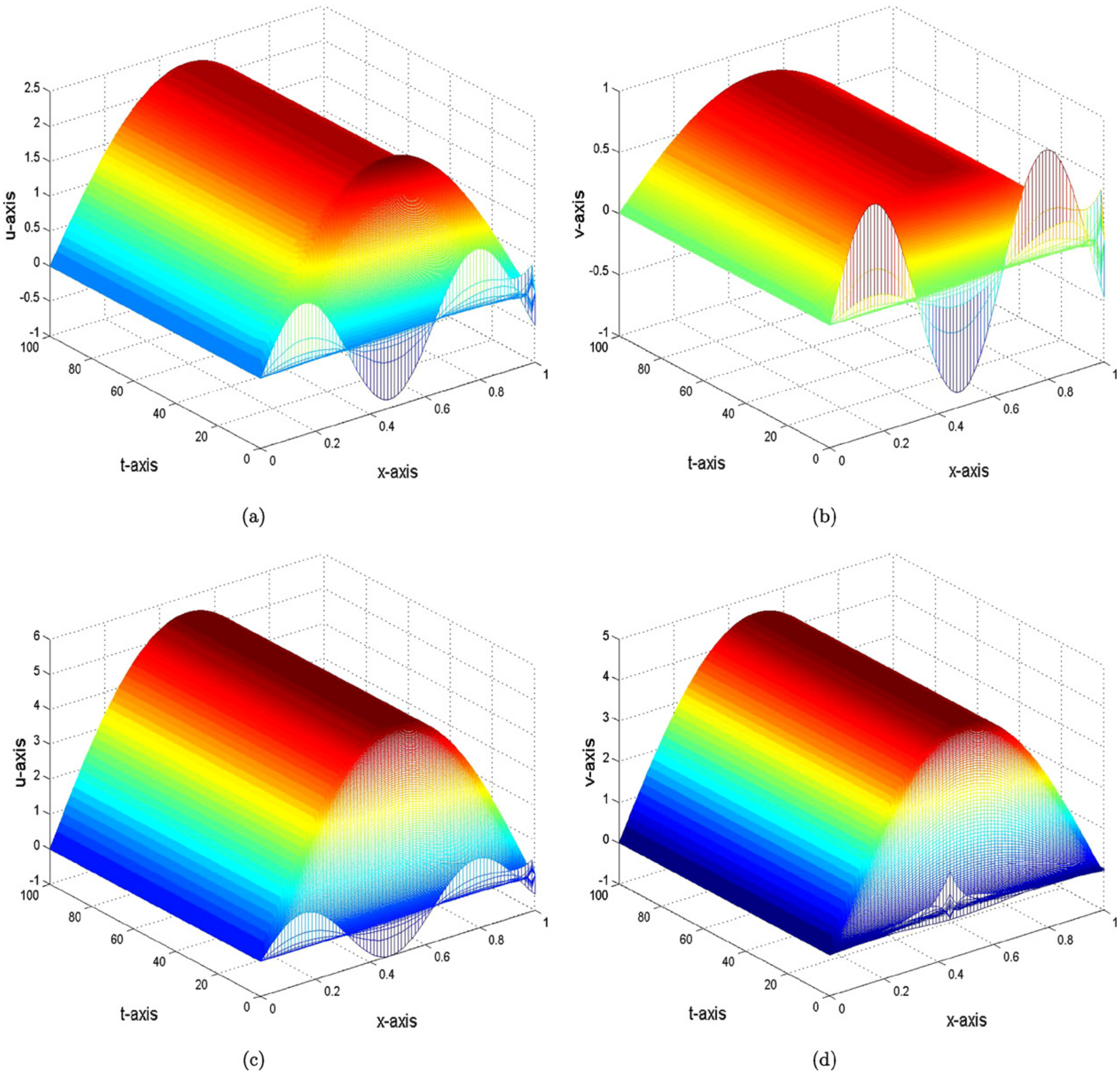

Fig. 1. Positive solutions (u, v) of (1.1). In the first row, $a = 12$, $c = 10$, $b = d = m = k = 1$. In the second row, $a = 15$, $c = 4$, $d = 20$, $b = m = k = 1$.

same way as in case (i). Therefore we may assume $\hat{u} \equiv 0$ by the maximum principle. Then we easily deduce that $\hat{v} = \theta_{\hat{c}}$. Hence $a = a' = \lambda_1(b\theta_{\hat{c}}/(1 + k\theta_{\hat{c}}))$, which is impossible since $a = \lambda_1(b\theta_{c_1}/(1 + k\theta_{c_1})) < \lambda_1(b\theta_{\hat{c}}/(1 + k\theta_{\hat{c}}))$. This completes the proof. $\square$

Remark 5. Theorem 3.6 gives a sufficient and necessary condition insuring that (1.2) has a positive solution when a is close to λ_1. But when a is bounded away from λ_1, $c < c_1$ is not a necessary condition any more. In fact, recalling Theorem 2.5 and Theorem 3.2 in [13], we find that there may be a positive solution when $a \leqslant \lambda_1(b\theta_c/(1 + k\theta_c))$. This implies that if $c \geqslant c_1$, then (1.2) has at least a positive solution under certain conditions.

4. Numerical simulation

In this section, we present some numerical simulations that verify and complement the analytic results in one dimension. All computations are performed with Matlab.

We may assume $\Omega = (0, 1)$. Then $\lambda_1 = \pi^2 \approx 9.870$. In most simulations performed, convergence to equilibrium was first observed. At the same time, our numerical simulation results illustrate the following major outcomes:

(1) Using Crank–Nicolson scheme, we described the positive solution of (1.1) in Fig. 1. Convergence to equilibrium was obviously observed. If predator can survive by itself in the absence of prey ($c = 10 > \lambda_1$), then prey and predator co-exist

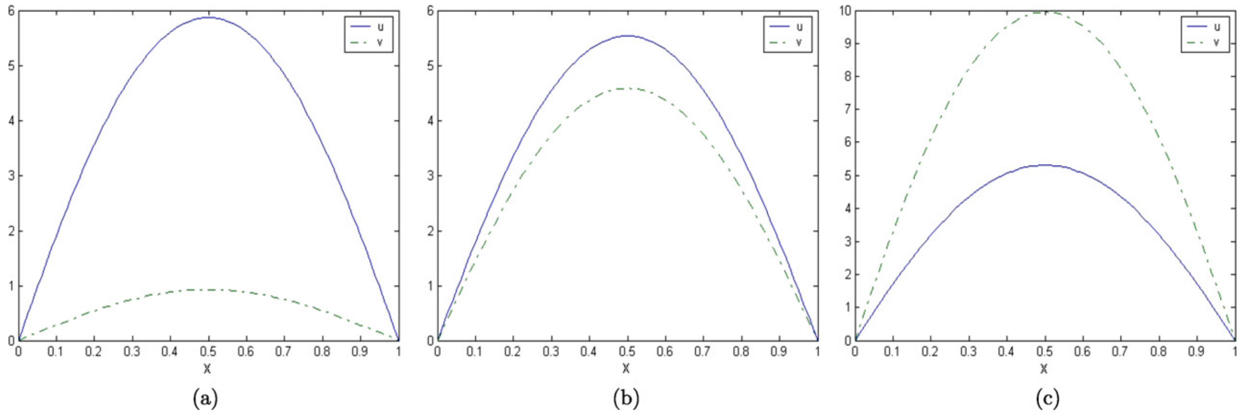

Fig. 2. Coexistence states (u, v) for different values of $c = -4, 4, 12$ with other parameters $a = 15$, $d = 20$, $b = m = k = 1$.

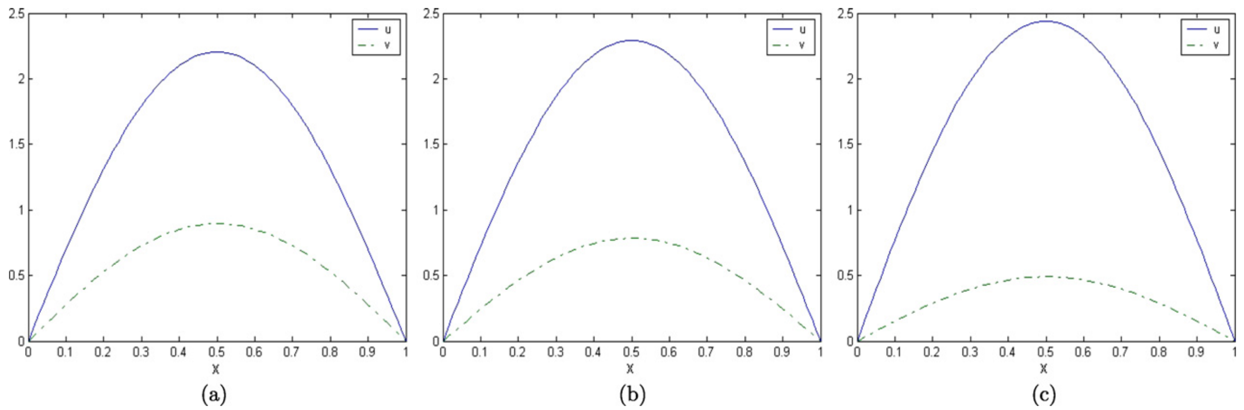

Fig. 3. Coexistence states (u, v) for different values of $k = 0.1, 10, 1000$ with other parameters $a = 12$, $c = 10$, $b = d = m = 1$.

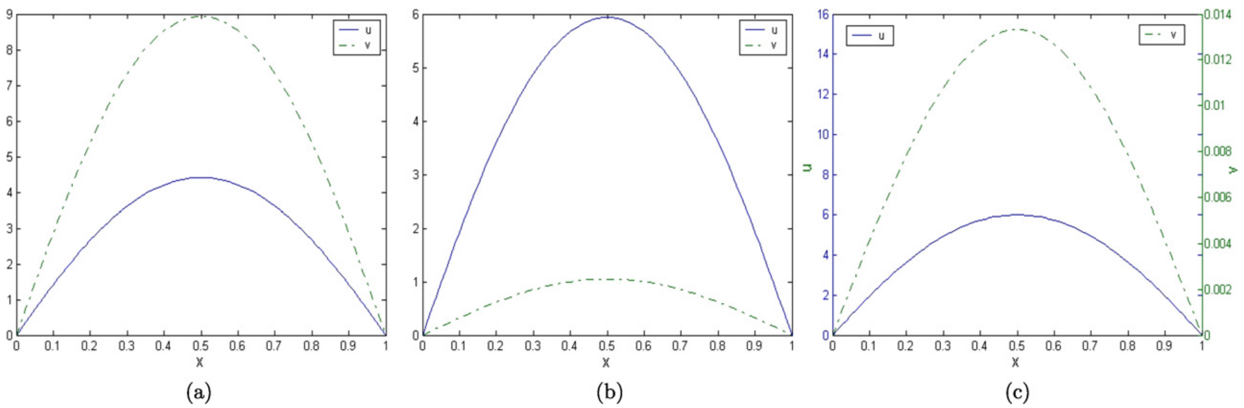

Fig. 4. Coexistence states (u, v) for different values of $k = 0.1, 10, 1000$ with other parameters $a = 15$, $c = 4$, $d = 20$, $b = m = 1$.

provided that the birth rate of prey is not too low ($a = 12 > \lambda_1(b\theta_c/(1 + k\theta_c)) \approx 9.985$), see Fig. 1(a) and (b). Another result shows that though the birth rate of predator is lower than that required to survive in the absence of prey ($c = 4 < \lambda_1$), two species can also co-exist, see Fig. 1(c) and (d). It should be noted that when the birth rate of predator or prey is too small, there is no coexistence state. In fact, plenty of numerical simulations suggest that the birth rate of prey must be larger than λ_1 and the birth rate of predator must be larger than $\lambda_1(-d\theta_a/(1 + m\theta_a))$.

(2) The effect of the parameter c on coexistence states was given in Fig. 2. We see that as c increases, the concentration of predator increases quickly, while the concentration of prey decreases slowly.

(3) The effect of k on coexistence states was described in Figs. 3 and 4. Fig. 3 deals with the case that $c > \lambda_1$, and Fig. 4 is for $c \leqslant \lambda_1$. Plenty of numerical simulations suggest that there is at most a unique and stable coexistence state for large k. Moreover, all results show that as k increases, the concentration of predator decreases, while the concentration of prey

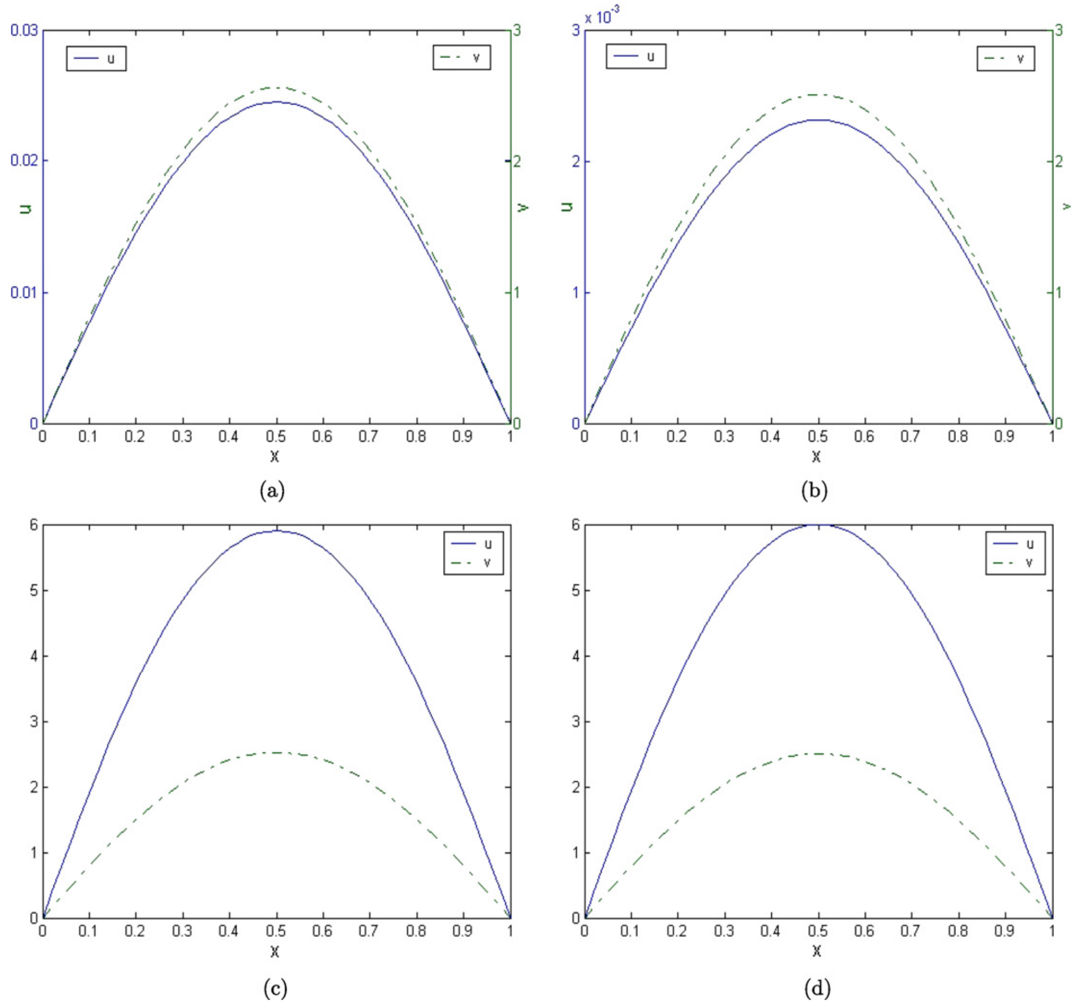

Fig. 5. Coexistence states (u, v) for different values of m with other parameters $a = 15$, $c = 12$, $b = 10$, $d = k = 1$. In the first column, $m = 50$, second one, $m = 5000$. Here, coexistence states in (c) and (d) are stable, and plenty of numerical examples strongly suggest coexistence states in (a) and (b) are unstable.

increases. But, the concentrations of two species in Fig. 4 change more obviously than that in Fig. 3. Furthermore, when $c > \lambda_1$ and k is large, the coexistence state is so close to (θ_a, θ_c), see Fig. 3(c); when $c \leqslant \lambda_1$ and k is large, the coexistence state is so close to $(\theta_a, 0)$ and kv is so close to a positive solution of (3.12), see Fig. 4(c).

(4) The effect of m on coexistence states was described in Fig. 5. If m is large, it seems there are only two coexistence states, one of which is close to $(0, \theta_c)$, see Fig. 5(a) for $m = 50$ and (b) for $m = 5000$, the other close to (θ_a, θ_c), see Fig. 5(c) for $m = 50$ and (d) for $m = 5000$. From Fig. 5(a) and (b), we find that the concentration of prey becomes lower and lower as m increases. The different case of m is just to show the asymptotic behavior of coexistence states. Moreover, we find that coexistence states in the second row of Fig. 5 are stable and these in the first row are (most possibly) unstable. These numerical results given in Fig. 5 verify the analytic results in [13].

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